Fourier Analysis   Feb 22, 2024

Review:

- We construct a cts function on the circle, say g, such that

$$S_N g(0) \not\to g(0)$$

Chap 4. Applications of Fourier series.

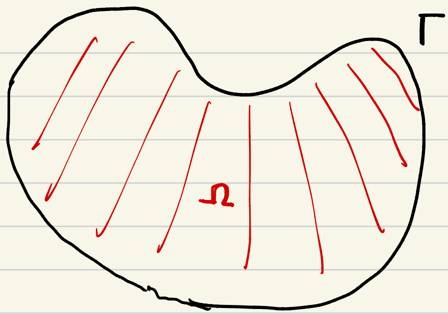
We are going to give the application of Fourier series in geometry, number theory, analysis and PDE.

§4.1 Isoperimetric inequality.

Thm 1. Let Γ be a C^1 simple closed curve in the plane. Let ℓ denote the length of Γ, let A denote the Area of the region bounded by Γ. Then

$$A \leq \frac{\ell^2}{4\pi}.$$

Moreover "=" holds if and only if Γ is a circle.



$A = \text{Area}(\Omega)$

$\ell = \text{length}(\Gamma)$

Then $A \leqslant \dfrac{\ell^2}{4\pi}$.

Def. A parametrized curve in $\mathbb{R}^2$ is a mapping

$$\gamma : [a,b] \to \mathbb{R}^2$$

The image of γ, $\{\gamma(t) : t \in [a,b]\}$, is called a curve, denoted by Γ.

Def. We say $\gamma = \gamma(t)$, $t \in [a,b]$, is C^1 if

$$t \mapsto \gamma(t) \text{ is } C^1 \text{ on } [a,b], \quad \gamma'(t) \neq 0 \text{ on } [a,b]$$

More precisely, writing $\gamma(t) = (x(t), y(t))$,

γ is C^1 $\Longleftrightarrow$ both $x(t)$ and $y(t)$ are C^1 on $[a,b]$.

and $(x'(t), y'(t)) \neq (0,0)$ on $[a,b]$.

Fact: The length of a C^1 parametrized curve γ is given by

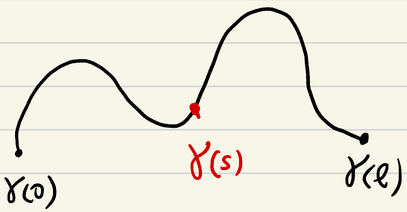
$$\text{length}(\Gamma) = \int_a^b |\gamma'(t)|\, dt$$

where $|\gamma'(t)| = \sqrt{x'(t)^2 + y'(t)^2}$

for $\gamma(t) = (x(t), y(t))$, $t \in [a,b]$.

Def. A parametrized curve $\gamma : [0, \ell] \to \mathbb{R}^2$ is said to be parametrized by arclength if

$$|\gamma'(t)| = 1 \quad \text{for} \quad 0 \leq t \leq \ell.$$



The length of the curve between $\gamma(0)$ and $\gamma(s)$

$$= \int_0^s |\gamma'(t)|\,dt = \int_0^s 1\, dt = s$$

It is not difficult to check that
any C^1 simple curve can be reparametrized by arclength.

Lem 2. Let f, g be integrable on the circle.

Then $$\langle f, g\rangle = \sum_{n=-\infty}^{\infty} \hat{f}(n)\, \overline{\hat{g}(n)}, \qquad (*)$$

where

$$\langle f, g\rangle = \frac{1}{2\pi}\int_0^{\pi} f(x)\,\overline{g(x)}\,dx.$$

Pf. This result is a generization of the Parseval identity.

Indeed if $f = g$, then $(*)$ becomes the Parserval identity

To prove $(*)$, let us use the following identity

$$\langle f, g\rangle = \frac{1}{4}\left(\|f+g\|^2 - \|f-g\|^2 + i\left(\|f+ig\|^2 - \|f-ig\|^2\right)\right)$$

which can be checked directly. Then by the Parseval identity

$$\langle f, g\rangle = \frac{1}{4}\left(\sum_{n=-\infty}^{\infty} |\hat{f}(n)+\hat{g}(n)|^2 - |\hat{f}(n)-\hat{g}(n)|^2 + i\left(|\hat{f}(n)+i\hat{g}(n)|^2 - |\hat{f}(n)-i\hat{g}(n)|^2\right)\right)$$

$$= \sum_{n=-\infty}^{\infty} \hat{f}(n) \overline{\hat{g}(n)} .$$

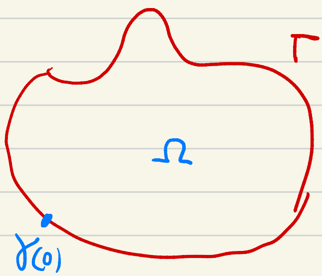
Proof of the isoperimetric inequality:

By taking a possible transformation

$$(x, y) \to (\delta x, \delta y)$$

if necessary, we may assume that

$$\ell(\Gamma) = 2\pi$$



Parametrize Γ by its arclength,

$$\gamma = \gamma(t) : \quad 0 \leq t \leq 2\pi$$

such that

$$|\gamma'(t)| = \sqrt{x'(t)^2 + y'(t)^2} = 1 \quad \text{for } t \in [0, 2\pi]$$

To estimate $A = \text{Area}(\Omega)$, let us use Green Thm:

Green Thm: For C^1 functions $P(x,y)$ and $Q(x,y)$,

$$\oint_\Gamma P(x,y)\,dx + Q(x,y)\,dy = \iint_\Omega \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dx\,dy$$

In this theorem, taking $P(x,y) \equiv 0$ and $Q(x,y) = x$ gives

$$\oint_\Gamma x\,dy = \iint_\Omega 1\,dx\,dy = \text{Area}(\Omega) = A.$$

That is,

$$A = \oint_\Gamma x\,dy = \int_0^{2\pi} x(t)\,y'(t)\,dt$$

Next we need to show that

$$A = \int_0^{2\pi} x(t)\,y'(t)\,dt \leq \frac{\ell^2}{4\pi} = \frac{(2\pi)^2}{4\pi} = \pi.$$

Under the condition $x'(t)^2 + y'(t)^2 = 1$.

For this purpose, we expand $x(t)$, $y(t)$ into their Fourier series on $[0, 2\pi]$.

$$x(t) = \sum_{n=-\infty}^{\infty} a_n e^{int}, \quad y(t) = \sum_{n=-\infty}^{\infty} b_n e^{int}.$$

(the above Fouriers converge because $x(t)$, $y(t)$ are diff)

$$x'(t) \sim \sum_{n=-\infty}^{\infty} in\, a_n e^{int}, \quad y'(t) \sim \sum in\, b_n e^{int}.$$

$$\left(\widehat{f'}(n) = in \hat{f}(n) \right)$$

By Parseval identity

$$\frac{1}{2\pi} \int_0^{2\pi} x'(t)^2 \, dt = \sum_{n=-\infty}^{\infty} |in\, a_n|^2 = \sum_{n=-\infty}^{\infty} n^2 |a_n|^2.$$

$$\frac{1}{2\pi} \int_0^{2\pi} y'(t)^2 \, dt = \sum_{n=-\infty}^{\infty} n^2 |b_n|^2.$$

Hence

$$1 = \frac{1}{2\pi} \int_0^{2\pi} x'(t)^2 + y'(t)^2 \, dt = \sum_{n=-\infty}^{\infty} n^2 \left(|a_n|^2 + |b_n|^2 \right).$$

(*)

Also by the generalized Parseval identity,

$$\frac{1}{2\pi}\int_0^{2\pi} x(t)\,y'(t)\,dt = \frac{1}{2\pi}\int_0^{2\pi} x(t)\,\overline{\overline{y'(t)}}\,dt$$

$$= \langle x(t), \overline{y'(t)} \rangle$$

$$= \sum_{n=-\infty}^{\infty} \hat{x}(n) \cdot \overline{\widehat{\overline{y'}}(n)}$$

$$= \sum_{n=-\infty}^{\infty} a_n \, \overline{in\, b_n}$$

$$= \sum_{n=-\infty}^{\infty} -in\, a_n \overline{b_n}.$$

Hence $A = \int_0^{2\pi} x(t)\,y'(t)\,dt = 2\pi \cdot \sum_{n=-\infty}^{\infty} \left(-in\, a_n \overline{b_n}\right).$

So

$$A = 2\pi \left| \sum_{n=-\infty}^{\infty} \left(-in\, a_n \overline{b_n}\right) \right|$$

$$\leq 2\pi \sum_{n=-\infty}^{\infty} |n|\,|a_n|\,|b_n|$$

$$\leq 2\pi \sum_{n=-\infty}^{\infty} |n| \frac{|a_n|^2+|b_n|^2}{2}$$

$$\leq 2\pi \cdot \sum_{n=-\infty}^{\infty} |n|^2 \frac{|a_n|^2+|b_n|^2}{2}$$

$$= \pi \quad \text{(by (*))}.$$

This proves the isoperimetric inequality.